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Husimi Distribution and the Fisher Information

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1. Introduction

In this chapter we review some aspects of the concept of the Fisher information measure in phase space for two specific systems: the Landau Diamagnetism and the Rigid rotator. The indispensable tool in this proposal is a quasi probability called Husimi distribution [1], which is frequently employed to characterize the quantum and classical behavior [2] of systems. Also, it possesses interesting applications in several areas of physics such as Quantum Mechanics, Quantum Optics, Information Theory and Nanotechnology [3–10]. Its main properties are: 1) it is definite positive in all phase space, 2) it possesses no correct marginal properties, 3) it permits to calculate the expectation values of observables in quantum mechanics similarly to the classical case [11], and 4) it is a special type of probability that simultaneously approximate location of position and momentum in phase space. It is important to note that this quasi probability is constructed by definition as the expectation value of the density operator in a basis of coherent states [12]. Details about the formulation of coherent states and the obtaining of Husimi distribution from these can be found on our chapter that it can be read in Ref. [13]. The main propose of this chapter is to present to the reader interesting problems in physics, such as, the harmonics oscillator [5], the Landau diamagnetism model [8, 14] and, the rigid rotator [7, 15], analyzed from a point of view of the information measures. In particular, we will put emphasis in the Fisher Information measure and its construction starting from a well-defined set of coherent states.

In our previous contribution published in Ref. [13] we research about a special semi-classical measure, the Wehrl entropy, as an important application of the Husimi distribution. In the present study we analyze some consequences of obtaining the Husimi distribution; for instance, the Fisher information for fundamental problems in physics for which the coherent states formulation is well defined.

In physics, great attention has been paid to the Landau diamagnetism which consists in a particle charged in a uniform magnetic field. For our purpose we will use a complete description of the Husimi Distribution in three dimensions in order to study such system, so as it was shown in our previous contributions (see Ref. [13], where we have discussed some limiting cases as high and low temperatures. From the present analysis, when three dimensions are considered, naturally arises a lower temperature

bound, whereby it is not possible to work in all finite temperatures. Such discussion is explained with details in Ref. [13].

The other system, that we take into account here, is the linear rigid rotator and its corresponding 3D anisotropic version. We analyze phase space delocalization and obtain the concomitant semiclassical Fisher information measure constructed by using Husimi Distribution constructed from suitable basis of coherent states.

In order to facilitate the understanding of this chapter to the reader, we give the following organization: in section 2 we begin introducing the concepts and methodology that will employ in the rest of the chapters. In section 3 we focus our attention on the Husimi distribution and the Fisher measure for the Landau diamagnetism. In section 4 we study the delocalization into phase space, within a semiclassical context by recourse to the Husimi distribution, for both cases of rigid rotators: linear and 3D–anisotropic instances. Finally, some conclusions and open problems are commented in section 5.

2. Previous concepts

This section provides reference material that we consider relevant to conveniently understand the development of this chapter. These are *i)* the Husimi distribution, *ii)* Wehrl entropy, and *iii)* Fisher information measure. In all cases, we refer to the model of the harmonic oscillator in a thermal state.

2.1. Husimi distribution and Wehrl entropy

From the standard statistical mechanics, the thermal density matrix can be represented by

$$\hat{\rho} = Z^{-1} e^{-\beta \hat{H}}, \quad (1)$$

where $\beta = 1/k_B T$ the inverse temperature T , and k_B the Boltzmann constant [16], \hat{H} is the Hamiltonian of the system and $Z = \text{Tr}(e^{-\beta \hat{H}})$ is the partition function.

In the current strategy, the expectation value of the density operator in a basis of coherent states is related to the Husimi distribution as [1]

$$\mu(z) = \langle z | \hat{\rho} | z \rangle, \quad (2)$$

where the set $\{|z\rangle\}$ denotes the eigenstates of the annihilation operator \hat{a} , i.e., $\hat{a}|z\rangle = z|z\rangle$ defined for all $z \in \mathbb{C}$ [12] and they are the coherent states for the system. Therefore, the normalization of the distribution is given by

$$\int \frac{d^2 z}{\pi} \mu(z) = 1, \quad (3)$$

where the integration is over the complex plane z and the element of integration is an area proportional to phase space element given by $d^2 z = dx dp / 2\hbar$.

The set $\{E_n\}$ stands for the spectrum of an arbitrary Hamiltonian \hat{H} , where n is a positive integer. With these elements the Husimi distribution takes the form

$$\mu(z) = \frac{1}{Z} \sum_n e^{-\beta E_n} |\langle z|n\rangle|^2, \quad (4)$$

where the set $\{|n\rangle\}$ represents energy eigenstates with eigenvalues E_n [4, 5].

A direct application that is additionally a useful measure of localization in phase-space [17, 18] is the Wehrl entropy, which is suitably defined as

$$W = - \int \frac{d^2z}{\pi} \mu(z) \ln \mu(z). \quad (5)$$

As a consequence of the uncertainty principle, Lieb [4] proved the inequality $W \geq 1$ which was previously conjectured by Wehrl [17].

For the Hamiltonian $\hat{H} = \hbar\omega[\hat{a}^\dagger \hat{a} + 1/2]$ of the harmonic oscillator, it is obtained a basis $\{|n\rangle\}$ and the spectrum $E_n = \hbar\omega(n + 1/2)$, with $n = 0, 1, \dots$ from the complete orthonormal set of eigenstates and eigenvectors, respectively. The algebra allows us to define the following elementary properties:

1. A set of Glauber coherent states is given by [19]

$$|z\rangle = e^{-|z|^2/2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} |n\rangle. \quad (6)$$

2. The normalization,

$$\langle n|n'\rangle = \delta_{n,n'} \quad (7)$$

where $\delta_{n,n'}$ is the Kronecker delta function.

3. The completeness property is contained in the relation

$$\sum_{n=0}^{\infty} |n'\rangle \langle n| = \hat{1}, \quad (8)$$

where $\hat{1}$ represents the identity operator in the defined space of eigenvectors.

Now, a suitable application of the present theoretical characterization [4] comes from certain calculations of the harmonic oscillator as the Husimi distribution

$$\mu_{HO}(z) = (1 - e^{-\beta\hbar\omega}) e^{-(1-e^{-\beta\hbar\omega})|z|^2}, \quad (9)$$

and the Wehrl entropy

$$W_{HO} = 1 - \ln(1 - e^{-\beta\hbar\omega}), \quad (10)$$

which are respectively known and useful analytical expressions [4].

2.2. Fisher information measure

A pertinent quantifier of information, which possess innumerable applications in several fields of Physics, is the Fisher information measure [20]. The last years have witnessed a great deal of activity revolving around physical applications of Fisher information measure [20, 21] providing tools to yield most of the canonical Lagrangians of theoretical physics [20, 21] related properly to the Boltzmann entropy [22, 23]. The Fisher information connected with translations of an observable x with the consistent probability density $\rho(x)$ is given by [24]

$$\mathcal{F} = \int dx \rho(x) \left[\frac{\partial \ln \rho(x)}{\partial x} \right]^2, \quad (11)$$

and the Cramer–Rao inequality is given by [24]

$$\Delta x \geq \mathcal{F}^{-1} \quad (12)$$

where Δx is the variance for the stochastic variable x which is of the form [24]

$$\Delta x^2 = \langle x^2 \rangle - \langle x \rangle^2 = \int dx \rho(x) x^2 - \left(\int dx \rho(x) x \right)^2. \quad (13)$$

In particular, it is interesting to study its representation appealing to the semiclassical approach (see, for example, Ref. [25] and references therein), whose main tool is a distribution function in phase space in the basis of coherent states. Specially, in this proposal, we pay attention to a particular distribution, the well-known Q –function or Husimi distribution.

An original, compact expression in phase space is advanced for the “semiclassical” Fisher information measure, that can be easily derived from the Wehrl-methodology described in Refs. [5] and [6]. The appearance for this measures reads

$$\mathcal{F} = \frac{1}{4} \int \frac{d^2 z}{\pi} \mu(z) \left\{ \frac{\partial \ln \mu(z)}{\partial |z|} \right\}^2, \quad (14)$$

which will be used in the following sections.

Inserting the μ –expression for the harmonic oscillator into Eq. (14) we find its analytical form

$$\mathcal{F}_{HO} = 1 - e^{-\beta \hbar \omega}, \quad (15)$$

leading to the following limits:

$$\begin{aligned} \text{for } T \rightarrow 0 \text{ one has } \mathcal{F}_{HO} &= 1 \\ \text{for } T \rightarrow \infty \text{ one has } \mathcal{F}_{HO} &= 0, \end{aligned} \quad (16)$$

as it should be expected.

3. Landau Diamagnetism: Charged particle in a uniform magnetic field

Diamagnetism is a problem firstly appointed by Landau who showed the discreteness of energy levels for a charged particle in a magnetic field [26]. By the observation of the diverse scenarios in the framework provided by the Landau diamagnetism we can study some relevant physical properties [27–29] as the role of the size of systems or the influence of boundaries, also the thermodynamic limit or quasi-stationary states. The primary motivation even today for several specialists is to find a useful measure to characterize theoretically every practical consequence of the system and its behavior.

In the past, Feldman and Kahn calculated the proper partition function for this system by appealing to the concept of Glauber coherent states from a set of basis states [30]. This formulation uses classical concepts as electron orbits, even though it contains all quantum effects [30]. This approach was previously used to obtain measures as the Wehrl entropy [17, 18] and Fisher information [31] with the purpose of studying the thermodynamics of the free spinless charged particle in a uniform magnetic field [32], this is the Landau diamagnetism problem. As observed, in such contribution the formulation is not completely consistent because it was necessary to normalize the Husimi distribution in order to arrive to reliable expressions for semiclassical measures [9, 32, 33].

Certainly, because the relevant effects seem to come only from the transverse motion, several efforts are made to describe this problem in two dimensions [9, 28, 29, 32–35]. Furthermore, the discovery of the quantum Hall effect has aroused much interest in understanding the behavior of electrons moving in a plane perpendicular to the magnetic field [35]. The confinement is possible at the *interface* typically between a semiconductor and an insulator, where a quantum well that traps the particles is formed, allowing their motion just in the direction of the interface plane at low energies, forbidding the motion in any other directions.

Conversely, we discuss here this problem in three dimensions, the most complete formulation. However, if the length of the cylindrical geometry of the system is large enough the results are close to those in two dimensions. Despite this latter, it is suggested that the formulation in two dimensions is not sufficient to explain the whole problem. As suggested before, electronic devices are based in interfaces. As a consequence of this line of reasoning, a natural lower temperature bound is theoretically imposed, that appears from the analysis in three dimensions.

3.1. The model of one charged particle in a magnetic field

We introduce the present application giving the essential ingredients of the well-known Landau model for diamagnetism: a spinless charged particle in a magnetic field B . Consider the kinetic momentum

$$\vec{\pi} = \vec{p} + \frac{q}{c} \vec{A}, \quad (17)$$

where m_q is the mass of a particle of charge q , the vector \vec{p} is the linear momentum subject to the action of \vec{A} , the vector potential.

If we follow the presentation of Feldman et al. [30]), the Hamiltonian reads [30]

$$H = \frac{\vec{\pi} \cdot \vec{\pi}}{2m_q}, \quad (18)$$

and the magnetic field is $\vec{B} = \vec{\nabla} \times \vec{A}$. The vector potential is chosen in the symmetric gauge as $\vec{A} = (-By/2, Bx/2, 0)$, which corresponds to a uniform magnetic field along the z -direction.

By using the formulation of the step-ladder operators [30], one needs to define the step operators as follows [30]

$$\hat{\pi}_{\pm} = \hat{p}_x \pm i\hat{p}_y \pm \frac{i\hbar}{2\ell_B^2}(\hat{x} \pm i\hat{y}), \quad (19)$$

where the length

$$\ell_B = (\hbar c/qB)^{1/2} \quad (20)$$

is the classical radius of the ground-state Landau orbit [30]. Motion along the z -axis is free [30]. For the transverse motion, the Hamiltonian specializes to [30]

$$\hat{H}_t = \frac{\hat{\pi}_+ \hat{\pi}_-}{2m_q} + \frac{1}{2} \hbar \Omega \hat{1}, \quad (21)$$

where an important quantity characterizes the problem, namely,

$$\Omega = qB/m_q c, \quad (22)$$

the cyclotron frequency [36]. The set of eigenstates $\{|N, m\rangle\}$ is characterized by two quantum numbers: N related to the energy and m with the z -projection of the angular momentum. They are consequently eigenstates of both \hat{H}_t , the Hamiltonian, and \hat{L}_z , the angular momentum operator [30], thus

$$\hat{H}_t |N, m\rangle = \left(N + \frac{1}{2}\right) \hbar \Omega |N, m\rangle = E_N |N, m\rangle \quad (23)$$

and

$$\hat{L}_z |N, m\rangle = m\hbar |N, m\rangle. \quad (24)$$

The eigenvalues of \hat{L}_z are not bounded by below, because m takes the values $-\infty, \dots, -1, 0, 1, \dots, N$ [30]. This fact agrees with the energies $(N + 1/2)\hbar\Omega$ that are infinitely degenerate [36]. As seen below, for estimation purposes, the physical relevance of phase-space localization is diminished by this fact. In addition, L_z is not an independent constant of the motion [36].

There exists a analogous formulation of an charged particle in a magnetic field by Kowalski that takes into account the geometry of a circle [33] (and for a comparison with the Feldman formulation see Ref. [9]), but at this point, we choose the Feldman formulation to work because the measure is easily defined and the normalization condition and other semiclassical measures are well described.

3.2. Husimi distribution and Wehrl entropy

We will start our present endeavor defining the Hamiltonian $\hat{H} = \hat{H}_t + \hat{H}_l$ where $\hat{H}_t = \hbar\Omega(\hat{N} + 1/2)$ to describe the transverse motion, being Ω the cyclotron frequency as defined by the Eq. (22) and \hat{N} the number operator; the Hamiltonian $\hat{H}_l = \hat{p}_z^2/2m_q$ to represent the longitudinal one-dimensional free motion, for a particle of mass m_q and charge q in a magnetic field B . A possible way to define the Husimi function η is given by

$$\eta(x, p_x; y, p_y; p_z) = \langle \alpha, \xi, k_z | \hat{\rho} | \alpha, \xi, k_z \rangle, \quad (25)$$

where $\hat{\rho}$ is the thermal density operator and the set $\{|\alpha, \xi, k_z\rangle\}$ stands for the coherent states for the description in three dimensions. By the direct product $|\alpha, \xi, k_z\rangle \equiv |\alpha, \xi\rangle \otimes |k_z\rangle$, the set $\{|\alpha, \xi\rangle\}$ corresponds to the coherent states of the transverse motion and $\{|k_z\rangle\}$ to the longitudinal motion. Therefore, the thermal density operator is given by

$$\hat{\rho} = \frac{1}{Z} e^{-\beta(\hat{H}_t + \hat{H}_l)}, \quad (26)$$

where $\beta = 1/k_B T$, T is the temperature and k_B the Boltzmann constant. In addition, Z is the partition function for motion in three dimensions of the particle. Now, if Z can be separated by using Z_t (the contribution for the transverse motion) and Z_l (the contribution for the one-dimensional free motion), then the partition function could be written as $Z = Z_l Z_t$. Thus, the Husimi function [1] is expressed as

$$\eta = \frac{e^{-\beta p_z^2/2m_q}}{Z_l Z_t} \sum_{n,m} e^{-\beta \hbar \Omega (n+1/2)} |\langle n, m | \alpha, \xi \rangle|^2. \quad (27)$$

where

$$Z_l = (\mathcal{L}/h)(2\pi m_q k_B T)^{1/2} \quad \text{and} \quad (28)$$

$$Z_t = \mathcal{A} m_q \Omega / (4\pi \hbar \sinh(\beta \hbar \Omega / 2)), \quad (29)$$

being \mathcal{L} the length of the cylinder, $\mathcal{A} = \pi R^2$ the area for cylindrical geometry [30]. In addition, the matrix element $|\langle n, m | \alpha, \xi \rangle|^2$ describes the probability of finding the particle in the coherent state $|\alpha, \xi\rangle$. Its expression was defined previously [37].

The distribution η is written as:

$$\eta = \eta_l(p_z) \eta_t(x, p_x; y, p_y), \quad (30)$$

where η has been separated as a function of two distributions, namely, $\eta_l = \eta_l(p_z)$ and $\eta_t = \eta_t(x, p_x; y, p_y)$. The explicit form of the Hamiltonian \hat{H}_l makes to miss the dependence on the variable z . Therefore, summing in Eq. (27) we solve

$$\eta_l = \frac{e^{-\beta p_z^2/2m_q}}{Z_l}, \quad (31)$$

$$\eta_t = \frac{2\pi\hbar}{\mathcal{A}m_q\Omega} (1 - e^{-\beta\hbar\Omega}) e^{-(1 - e^{-\beta\hbar\Omega})|\alpha|^2/2\ell_B^2}, \quad (32)$$

where the length ℓ_B is defined by the Eq. (20). From expressions (31) and (32), we emphasize again that $\eta_l(p_z)$ describes the free motion of the particle in the magnetic field direction and $\eta_t(x, p_x; y, p_y)$ the Landau levels due to the circular motion in a transverse plane to the magnetic field, similar to the harmonic oscillator of Eq. (9) since $|z|^2 \rightarrow |\alpha|^2/2\ell_B^2$. Consequently Eqs. (30), (31) and (32) together contain the complete description of the system. We noticed both distributions are naturally normalized in a standard form, i.e.,

$$\int \frac{dz dp_z}{h} \eta_l(p_z) = 1, \quad (33)$$

and

$$\int \frac{d^2\alpha d^2\xi}{4\pi^2\ell_B^4} \eta_t(x, p_x; y, p_y) = 1. \quad (34)$$

In consequence, both Eqs. (31) and (32), under conditions (33) and (34), allow us to get a close form for the Wehrl entropy. Furthermore, using one of the most basic property of the entropy, the additivity, we can state $W_{\text{total}} = W_l + W_t$. Hence,

$$W_l = - \int \frac{dz dp_z}{h} \eta_l(p_z) \ln \eta_l(p_z), \quad (35)$$

$$W_t = - \int \frac{d^2\alpha d^2\xi}{4\pi^2\ell_B^4} \eta_t(x, p_x; y, p_y) \ln \eta_t(x, p_x; y, p_y), \quad (36)$$

again, the subindexes t and l represent respectively the transverse and longitudinal motions.

As a consequence of solving the integrals (35) and (36) we can identify the two entropies, they are

$$W_l = \frac{1}{2} + \ln \left(\frac{\mathcal{L}}{\lambda} \right), \quad (37)$$

$$W_t = 1 - \ln \left(1 - e^{-\beta\hbar\Omega} \right) + \ln(g), \quad (38)$$

where $\lambda = h/(2\pi m_q k_B T)^{1/2}$ represents the mean thermal length of the particle and $g = \mathcal{A}/2\pi\ell_B^2$ the degeneracy of a Landau level [38].

3.3. Semiclassical behavior

In fact, the classical entropy for a free particle in one dimension and Eq. (37) are coincident. Furthermore, the Eq. (38) is the Wehrl entropy for the transverse motion and possesses a form close to the harmonic oscillator entropy given by the Eq. (10), with the exception of a term associated with the degeneracy. Some properties of entropies that can be directly derived from Eqs. (37) and (38) are:

1. As commented before, W_l and the classical entropy for the free motion in one dimension coincide between them. Furthermore, this part of the entropy has to be nonnegative at all temperatures, this is $W_l \geq 0$. This condition imposes a minimum to the temperature, given by

$$T_0 = \frac{h^2}{2\pi m_e e k_B \mathcal{L}^2}, \quad (39)$$

where $e = 2.718281828$. Due to this basic property of W_l , the system is forced to take high values of temperature, being $T > T_0$, where the behavior of the system is classical. Equivalently, it is possible to assure that, if $T/T_0 \geq 1$, the length of a thermal wave λ lower than the average of the spacing among particles and quantum considerations are not relevant [39]. In addition, T_0 does not depend on external or internal physical parameters related to the system, as the transverse area, external magnetic field, charge of the particle, etc, practically depends only on the size of the system. If the system is large enough, the minimum temperature is low. However, modern electronic systems possess junctions where \mathcal{L} can be considered almost zero. Thus, minimum temperature required to make applicable the present description is enough high [40].

2. The Wehrl entropy that is associated with transverse motion satisfies $W_t \geq 1 + \ln(g)$ for all temperatures of the system, which is very nearly the Lieb condition in one dimension [41] with an additional term given by the logarithm of g , the degeneracy. The transverse motion is approximately bi-dimensional, but the Landau approach reduces the quantum motion of the particle in a magnetic field to a degenerate spectrum in one dimension essentially recovering the physics of the missing dimension. Therefore, the discussion about the behavior of the Wehrl entropy in light of the Lieb condition does not increase any applicability of the present treatment because the latter is always satisfied. The main problem that appears from the emphasis on the transverse motion is the restricted vision that is obtained of the behavior of the system. [9, 30, 32, 33], which represents the main difference with other contributions that discuss this topic. The combination of reasoning including both motions has sense when the imposition over the temperature is satisfied. For values of the temperature lower than T_0 , the behavior is essentially anomalous, thus this proposal is not applicable.

Additionally, the total entropy is expressed simply as follows

$$W_{\text{total}} = \frac{3}{2} - \ln(1 - e^{-\beta \hbar \Omega}) + \ln(g) + \ln\left(\frac{\mathcal{L}}{\lambda}\right). \quad (40)$$

Now, we can discuss some approximate and limiting cases.

In first order of approximation, for $k_B T \gg \hbar \Omega$, we have $\ln(g/(1 - e^{-\beta \hbar \Omega})) \approx \ln(\mathcal{A}T/T_0 \mathcal{L}^2)$. If we write the thermal wave length in terms of the temperature T_0 , as $\lambda = \mathcal{L}(eT_0/T)^{1/2}$ and considering that $\mathcal{V} = \mathcal{A}\mathcal{L}$, the entropy (40) is rephrased as follows

$$W_{\text{total}}^{(1)} = \frac{3}{2} + \ln\left(\frac{\mathcal{V}}{\lambda^3}\right). \quad (41)$$

This is a particular expression for the entropy of a free particle in three dimensions related to the motion of a charged particle into a region of the magnetic field making mention of some geometrical properties of the system.

In second order of approximation, considering the special condition $\mathcal{A} \sim \mathcal{L}^2$, Wehrl entropy is expressed as follows

$$W_{\text{total}}^{(2)} \approx W_{\text{total}}^{(1)} + \frac{T_0}{T} g. \quad (42)$$

As explained before, the Wehrl entropy takes values that are permitted by the Lieb condition, namely, $W \geq 1$. According to Eq. (42) the slope decreases as temperature increases. This fact also illustrates why the disorder increases as the magnetic field increases too.

The lower bound of temperature is related to values of T greater than T_0 , because this approach does not consider any temperature less than T_0 . In addition to this, the behavior of the total Wehrl entropy is reduced to the logarithm of the magnetic field. In order to see what occurs in the limiting case of the lowest temperature, according to Eq. (39), we take systems with $L \rightarrow \infty$; thus the transverse entropy of Eq. (38) is rewritten as follows

$$W_t^{T \rightarrow 0^+} = 1 + \ln(g). \quad (43)$$

As aforementioned, the Wehrl entropy is similar to the entropy of the harmonic oscillator and the lowest temperature comes being greater than the bound temperature, thus $W \geq 1$ [41] as it was conjectured by Wehrl and shown by Lieb. From this condition, it must arrive to the following inequality for the magnetic field

$$g \geq 1, \quad (44)$$

where $g = q\mathcal{A}B/hc$ also accounts for the ratio between the flux of the magnetic field $\mathcal{A}B$ and the quantum of the magnetic flux given by $hc/q = 4.14 \times 10^{-7} [\text{gauss cm}^2]$ [14]. Then the inequality (44) adopts the form

$$B \geq \frac{1}{\mathcal{A}} \frac{hc}{q} = B_0. \quad (45)$$

Moreover, the magnetic field $B_0 = hc/\mathcal{A}q$ becomes to take a bound limiting value representing a minimum value for the external magnetic field. If $\mathcal{A} \rightarrow \infty$, we can study what occurs to the system when the magnetic field close to zero.

Now, we add two comments about the quantum description of particles in magnetic field close to limiting values of temperatures and magnetic fields, respectively:

1. The quantum Hall effect is observed in two-dimensional electron systems subjected to low temperatures and strong magnetic fields and emerges from the Landau quantization [42, 43] which corresponds to a quantum version of the Hall effect [35]. The degeneracy is given by [14]

$$\phi = \nu \phi_0, \quad (46)$$

where $\phi_0 = hc/q$ is the minimum quantity (or quantum) of the magnetic flux. The factor ν takes integer values as $\nu = 1, 2, 3, \dots$ and it is related to the “*filling factor*” and simply with the conductivity quantization as $\sigma = \nu q^2/h$. The subsequent discovery of the fractional quantum Hall effect [34] expand the values ν to rational fractions as $\nu = 1/3, 1/5, 5/2, 12/5, \dots$. Thus the fractional quantum Hall effect relies on other phenomena associated with interactions. In any case, the degeneracy is ν greater than 1 due to the inequality (44), as before, the transverse entropy always satisfies the Lieb bound for all temperatures and large enough systems, obtaining an infinite family of Wehrl entropies

$$W_t = 1 - \ln(1 - e^{\beta \hbar \Omega}) + \ln \nu. \quad (47)$$

The limiting value of ν provides a good descriptor for the integer quantum Hall effect. Conversely, for the fractional values of ν less than 1 are left out the present approach.

2. The Haas-van Alphen effect is other phenomenon that we can discuss. It is observed at low enough values of temperatures, describing oscillations in the magnetization, because the particles tend to occupy the lowest energy states. In the present description it is manifest for finite values of \mathcal{A} and B lower than B_0 . Whereas if the value of the magnetic field decreases a less number of particles can be in the lowest state due to degeneracy is directly proportional to B [38]. Then, the transverse Wehrl entropy W_t is well defined for values of the magnetic field over B_0 , this is $B/B_0 \geq 1$ and/or $g \rightarrow 1^+$.

3.4. Fisher Information Measure versus degeneracy

In the present subsection we propose a compact expression for the transverse Fisher information measure, taking into account a special way formerly developed in Ref. [6], which is given by

$$\mathcal{F}_t = \int \frac{d^2\alpha d^2\xi}{4\pi^2 \ell_H^4} \eta_t(\alpha) \left(\frac{\partial \ln \eta_t(\alpha)}{\partial \alpha} \right)^2. \quad (48)$$

After introducing the known expression for η_t , we arrive to

$$\mathcal{F}_t = \frac{2}{\ell_H^2} (1 - e^{-\beta \hbar \Omega}). \quad (49)$$

Fisher measure \mathcal{F}_t has space dimension $(L)^{-2}$ and quantifies the ability for estimating the parameter α [44]. This parameter corresponds to the radio of a circular orbit of coherent states. By combining Eqs. (49) and (46) with the definition of ℓ_H we obtain

$$\mathcal{F}_t = \frac{4\pi\nu}{A} (1 - e^{-\beta \hbar \Omega}), \quad (50)$$

which represents the linear dependence of the measure \mathcal{F}_l with the magnetic field through the constant ℓ_H^2 at low temperature.

The inverse exponential dependence on the temperature, of the Fisher information, is clear from Eq. (50). Further, the initial value directly depends on the factor v .

Now, to complete the description of the movement, we consider the Fisher information measure for the longitudinal motion, this is

$$\mathcal{F}_l = \int \frac{dz dp_z}{h} \eta_l(p_z) \left(\frac{\partial \ln \eta_l(p_z)}{\partial p_z} \right)^2, \quad (51)$$

where p_z is the variable that we contain in the present discussion, which was previously ignored [32], making a great difference when the results are compared. The function η_l is included into the above equation to get

$$\mathcal{F}_l = \frac{\beta}{m}. \quad (52)$$

As seen before, the Fisher measure in one dimension coincides with the classical one for the free particle [45]. As expected, the total Fisher measure is constructed multiplying Eqs. (50) and (52).

3.5. Additional appointments and consequences

The Wehrl entropy, which we obtain here, depends on multiple parameters, for instance, the degeneracy g , and the ratio between the cylinder and thermal lengths, *i.e.*, \mathcal{L} and λ . The combination of these parameters can effectively give some interesting results. Therefore, in a especial perspective we can see that the harmonic oscillator is behaved as a particular case of the charged particle in a magnetic field. Thus, we can consider, for example, the following relation among parameters:

$$g = \frac{\lambda}{\mathcal{L}} \exp\left(\frac{1}{2}\right), \quad (53)$$

which leads the Wehrl entropy from the Landau diamagnetism to the one-dimensional harmonic oscillator (15). This is a nontrivial approach because the nature of problems are radically different. For instance, the harmonic oscillator, that we use here, is a one-dimensional system, but the Landau diamagnetism is three-dimensional. Consequently, phase spaces are not coincident and measures are not the same.

Besides, we know $g \geq 1$. But, if we consider the minimum value $g = 1$, we can obtain the bound value of the temperature T_0 , given by Eq. (39), above this value, the present approach is valid. Afterward, we obtain a relationship between both lengths involved into the problem, this is a bound value for the length of the cylinder, $\mathcal{L} \geq \lambda/e$. Thus, for values where this condition is violated, this approach is not valid.

The comparison between the Fisher information measures, for both cited problems, is also possible. Hence, in the same previous line we can propose a comparison of the Fisher information measures, considering measures dimensionally compatible. Originally, the classical Fisher information (11) accounts the localization of the corresponding probability density $\rho(x)$, which is approached by

Cramer-Rao inequality (12), where Δx is the variance for the stochastic variable x . However, the variation of the definition (11) takes into account the localization, not in the variable x or any other coordinate, but the localization into phase space. This is well defined for the transverse motion. Moreover, the longitudinal motion is classical, not quantized, and any coherent state formulation is proposed. The quantum counterpart can be defined as a problem of continuous spectrum [46], and a suitable formulation of coherent states is still unknown; for the time, this continues being an open problem. Thus, the classical formulation is used and we have decided to advance evaluating the classical distribution for the longitudinal motion.

In addition, with the purpose of describing the complete motion, we consider now the Fisher information measure for the movement, this is

$$\mathcal{F}' = \frac{\lambda^2}{\ell_H^2} (1 - e^{-\beta \hbar \Omega}). \quad (54)$$

where \mathcal{F}' is defined as $\mathcal{F}' = \hbar^2 I_t I_l / 4\pi$ in order to compare the trend of this Fisher measure with corresponding one of the harmonic oscillator. These cases are comparable with the harmonic oscillator only if $\lambda^2 = \ell_H^2$ and are depicted with red-solid-line in Fig. 1.

4. Description of the molecular rotation: Rigid rotator

There are few physical systems whose spectrum is analytically known, aside from the previous one we have the anisotropic rigid rotator, which is a system of a single particle that can rotate in several ways. Thermodynamic properties can be analytically described [47]. It is expected that this treatment can characterize important features of molecular systems [48] to apply such concepts to several aspects related to materials [49].

4.1. Linear rigid rotator

We begin exploring the linear rigid rotator based on the excellent discussion made in Ref. [50] about the coherent states for angular momenta. The Hamiltonian of this simple system is [16]

$$\hat{H} = \frac{\hat{L}^2}{2I_{xy}}, \quad (55)$$

where the operator \hat{L}^2 is associated with the angular momentum and the parameter I_{xy} is the corresponding inertia momentum. The set $\{|IK\rangle\}$ is the set of eigenstates of the Hamiltonian, where we can verify the following relations

$$\begin{aligned} \hat{L}^2 |IK\rangle &= I(I+1)\hbar^2 |IK\rangle \\ \hat{L}_z |IK\rangle &= K\hbar |IK\rangle, \end{aligned} \quad (56)$$

with $I = 0, 1, 2, \dots$, for $-I \leq K \leq I$. Additionally, the energy spectrum is given by eigenstates of the operator H

$$\varepsilon_I = \frac{I(I+1)\hbar^2}{2I_{xy}}. \quad (57)$$

A suitable construction of coherent states is found in Ref. [51, 52] for the lineal rigid rotator, using Schwinger oscillator model of angular momentum, in the fashion

$$|IK\rangle = \frac{(\hat{a}_+^\dagger)^{I+K}(\hat{a}_-^\dagger)^{I-K}}{\sqrt{(I+K)!(I-K)!}}|0\rangle, \quad (58)$$

where \hat{a}_+ , \hat{a}_- are the corresponding creation and annihilation operators, respectively, and they show the following basic properties

1. The vacuum state

$$|0\rangle \equiv |0,0\rangle$$

2. Orthogonality is satisfied by

$$\langle I'K'|IK\rangle = \delta_{I',I}\delta_{K',K},$$

3. The completeness property is contained in the relation

$$\sum_{I=0}^{\infty} \sum_{K=-I}^I |IK\rangle \langle IK| = \hat{1}.$$

Due to we are interested in two degrees of freedom, the resulting coherent states come from the tensor product of $|z_1\rangle$ and $|z_2\rangle$ [50, 53], where

$$|z_1 z_2\rangle = |z_1\rangle \otimes |z_2\rangle, \quad (59)$$

and

$$\hat{a}_+|z_1 z_2\rangle = z_1|z_1 z_2\rangle, \quad (60)$$

$$\hat{a}_-|z_1 z_2\rangle = z_2|z_1 z_2\rangle. \quad (61)$$

Therefore, $|z_1 z_2\rangle$ is the coherent state written [50] as

$$|z_1 z_2\rangle = e^{-\frac{|z|^2}{2}} e^{z_1 \hat{a}_+^\dagger} e^{z_2 \hat{a}_-^\dagger} |0\rangle, \quad (62)$$

with

$$|z_1\rangle = e^{-\frac{|z_1|^2}{2}} e^{z_1 \hat{a}_+^\dagger} |0\rangle, \quad (63)$$

$$|z_2\rangle = e^{-\frac{|z_2|^2}{2}} e^{z_2 \hat{a}_-^\dagger} |0\rangle. \quad (64)$$

We need to introduce the suitable notation

$$|z|^2 = |z_1|^2 + |z_2|^2. \quad (65)$$

Using Eqs. (58) and (62) we easily calculate $|z_1 z_2\rangle$ and, after a bit of algebra, find

$$|z_1 z_2\rangle = e^{-\frac{|z|^2}{2}} \sum_{n_+, n_-} \frac{z_1^{n_+}}{\sqrt{n_+!}} \frac{z_2^{n_-}}{\sqrt{n_-!}} |IK\rangle, \quad (66)$$

where $n_+ = I + K$ and $n_- = I - K$. Thus, the probability of obtaining the state $|IK\rangle$ in the coherent state $|z_1 z_2\rangle$ is of the form

$$|\langle IK | z_1 z_2 \rangle|^2 = e^{-|z|^2} \frac{|z_1|^{2n_+}}{n_+!} \frac{|z_2|^{2n_-}}{n_-!}. \quad (67)$$

The present coherent states satisfy resolution of unity

$$\int \frac{d^2 z_1}{\pi} \frac{d^2 z_2}{\pi} |z_1 z_2\rangle \langle z_1 z_2| = 1. \quad (68)$$

Furthermore, z_1 and z_2 are continuous variables.

The procedure developed by Anderson *et al.* [4] is easily followed and used to assess the Husimi distribution [1]. In our approach this is defined, from Eq. (4), as

$$\mu(z_1, z_2) = \langle z_1, z_2 | \hat{\rho} | z_1, z_2 \rangle, \quad (69)$$

where the density operator is

$$\hat{\rho} = Z_{2D}^{-1} \exp(-\beta \hat{H}). \quad (70)$$

The concomitant rotational partition function Z_{2D} is given in Ref. [16]

$$Z_{2D} = \sum_{I=0}^{\infty} (2I+1) e^{-I(I+1) \frac{\Theta}{T}}, \quad (71)$$

with $\Theta = \hbar^2 / (2I_{xy} k_B)$. We emphasize that in the present context the performing of the sum $\text{Tr} \equiv \sum_{I=0}^{\infty} \sum_{K=-I}^I$ corresponds to the “operation trace”. Using now the completeness property into Eq. (69) and the Eq. (67), we obtain the Husimi distribution in the form

$$\mu(z_1, z_2) = e^{-|z|^2} \frac{\sum_{I=0}^{\infty} \frac{|z|^{4I}}{(2I)!} e^{-I(I+1) \frac{\Theta}{T}}}{\sum_{I=0}^{\infty} (2I+1) e^{-I(I+1) \frac{\Theta}{T}}}. \quad (72)$$

It is easy to show that this distribution is normalized to unity

$$\int \frac{d^2 z_1}{\pi} \frac{d^2 z_2}{\pi} \mu(z_1, z_2) = 1, \quad (73)$$

where z_1 and z_2 are given by Eqs. (60), (61), and (65). We must employ the binomial expression $(|z_1|^2 + |z_2|^2)^{4I}$ and then integrate over the whole complex plane in two dimensions to verify the normalization condition. The differential element of area in the $z_1(z_2)$ plane is $d^2 z_1 = dx dp_x / 2\hbar$ ($d^2 z_2 = dy dp_y / 2\hbar$) [19]. Moreover, we have the phase-space relationships

$$|z_1|^2 = \frac{1}{4} \left(\frac{x^2}{\sigma_x^2} + \frac{p_x^2}{\sigma_{p_x}^2} \right), \quad (74)$$

$$|z_2|^2 = \frac{1}{4} \left(\frac{y^2}{\sigma_y^2} + \frac{p_y^2}{\sigma_{p_y}^2} \right), \quad (75)$$

where $\sigma_x \equiv \sigma_y = \sqrt{\hbar/2m\omega}$ and $\sigma_{p_x} \equiv \sigma_{p_y} = \sqrt{m\omega\hbar/2}$.

The profile of the Husimi function is similar to that of a Gaussian distribution.

As before, a semiclassical measure of localization is the Wehrl entropy [17], and the Fisher [5] as well. For the present model in two dimensions, the Wehrl entropy reads

$$\mathcal{W} = - \int \frac{d^2 z_1}{\pi} \frac{d^2 z_2}{\pi} \mu(z_1, z_2) \ln \mu(z_1, z_2), \quad (76)$$

where $\mu(z_1, z_2)$ is given by Eq. (72).

4.1.1. Fisher information measure

The Fisher measure [5, 20, 21] regards as a semiclassical counterpart of Wehrl entropy [5]. Now, extending the ideas developed in Ref. [5] for the case of the harmonic oscillator in one dimension to the present case in two dimensions, we can define the shift invariant Fisher measure in the fashion

$$\mathcal{F}_{2D} = \frac{1}{4} \int \frac{d^2 z_1}{\pi} \frac{d^2 z_2}{\pi} \mu(z_1, z_2) \left(\frac{\partial \ln \mu(z_1, z_2)}{\partial |z|} \right)^2. \quad (77)$$

From Eq. (72) it is easy to prove that

$$\eta(z_1, z_2) = \frac{1}{2} \frac{\partial \ln \mu(z_1, z_2)}{\partial |z|} = \frac{\sum_{I=0}^{\infty} \left[\frac{|z|^{4I-1}}{(2I-1)!} - \frac{|z|^{4I+1}}{(2I)!} \right] e^{-I(I+1)\Theta/T}}{\sum_{I=0}^{\infty} \frac{|z|^{4I}}{(2I)!} e^{-I(I+1)\Theta/T}}. \quad (78)$$

Therefore, the corresponding Fisher measure acquires the simpler appearance

$$\mathcal{F}_{2D} = \int \frac{d^2 z_1}{\pi} \frac{d^2 z_2}{\pi} \mu(z_1, z_2) \eta(z_1, z_2)^2, \quad (79)$$

i.e.,

$$\mathcal{F}_{2D} \equiv \langle \eta(z_1, z_2)^2 \rangle, \quad (80)$$

where with the notation

$$\langle \mathcal{G} \rangle = \int \frac{d^2 z_1}{\pi} \frac{d^2 z_2}{\pi} \mu(z) \mathcal{G}, \quad (81)$$

we refer to the *semi-classical expectation value* of \mathcal{G} . In Fig. 1 we plot the Fisher information and the Wehrl entropy as a function of the temperature (black-dashed-line), which we compare with the same measures for the transverse Landau diamagnetism (blue-solid-line). At low temperatures, the Fisher information measure describes the inverse-delocalization and takes its maximum value when the Wehrl entropy is minimum. This behavior is reversed for high temperatures. Every curve can be compared with the respective counterpart shown for the harmonic oscillator in one dimension (red-solid-line).

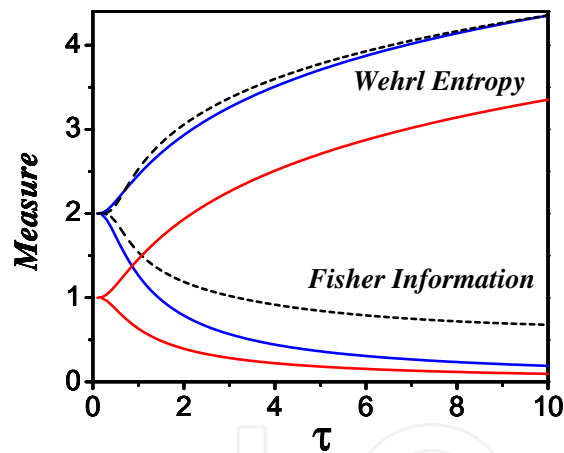


Figure 1. Trends of Fisher Information and Wehrl entropy for the rotator (black-dashed-line) in two dimensions is compared with the transverse Landau diamagnetism (blue-solid-line), the horizontal axis is the normalized temperature $\tau = kT(2I_{xy})/\hbar^2$ and $\tau = kT/\hbar\Omega$, respectively. Additionally, we show a case where the Landau diamagnetism dimensionally coincides with the one-dimensional harmonic oscillator (red-solid-line). The Wehrl entropy starts in $W=1$. If the normalized temperature increases, the Fisher information decreases while Wehrl entropy increases.

4.2. Rigid rotator in three dimensions

In the present section we consider a more general problem, the model of the rigid rotator in three dimensions, whose Hamiltonian writes [54]

$$\hat{H} = \frac{\hat{L}_x^2}{2I_x} + \frac{\hat{L}_y^2}{2I_y} + \frac{\hat{L}_z^2}{2I_z}, \quad (82)$$

where the parameters I_x , I_y , and I_z are the inertia momenta. The set $\{|IMK\rangle\}$ corresponds to a complete set of eigenvectors of the operator \hat{H} . The following relations are additionally applied

$$\begin{aligned}\hat{L}^2|IMK\rangle &= I(I+1)\hbar^2|IMK\rangle \\ \hat{L}_z|IMK\rangle &= K\hbar|IMK\rangle \\ \hat{J}_z|IMK\rangle &= M\hbar|IMK\rangle,\end{aligned}\tag{83}$$

with $-I \leq K \leq I$ and $-I \leq M \leq I$, where $I = 0, \dots, \infty$. The elements of set $\{|IMK\rangle\}$ satisfy orthogonality and completeness property [54]

$$\langle I'M'K'|IMK\rangle = \delta_{I',I}\delta_{M',M}\delta_{K',K}\tag{84}$$

$$\sum_{I=0}^{\infty} \sum_{M=-I}^I \sum_{K=-I}^I |IMK\rangle \langle IMK| = \hat{1}.\tag{85}$$

If we take $\hat{L}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2$ and assume axial symmetry, i.e., $I_{xy} \equiv I_x = I_y$, we can recast the Hamiltonian as

$$\hat{H} = \frac{1}{2I_{xy}} \left[\hat{L}^2 + \left(\frac{I_{xy}}{I_z} - 1 \right) \hat{L}_z^2 \right],\tag{86}$$

where the operator \hat{L}_z represents the projection on the rotation axis z of the \hat{L}^2 , which is the angular momentum operator. The concomitant spectrum of energy becomes

$$\varepsilon_{I,K} = \frac{\hbar^2}{2I_{xy}} \left[I(I+1) + \left(\frac{I_{xy}}{I_z} - 1 \right) K^2 \right],\tag{87}$$

where the number I is integer and non-negative and it stands for the eigenvalue of the operator \hat{L}^2 , the angular momentum. The range of the other quantum number $-I \leq m \leq I$ represents the projections on the intrinsic rotation axis of the rotator. Every state has a degeneracy $(2I+1)$. The inertia momenta are quantified by the parameters $I_x = I_y \equiv I_{xy}$ and I_z . The ratio I_{xy}/I_z characterizes different “geometrical” issues. For instance, some typical values of I_{xy}/I_z are 1, 1/2 and ∞ , which correspond to the spherical, the extremely oblate and prolate cases, respectively.

4.2.1. Construction of coherent states

Again, we cite the work of Morales *et al.* where they construct a suitable set of coherent states for the rigid rotator in Ref. [54] and kindly discuss their mathematical foundations. First, they start introducing the auxiliary quantity

$$X_{I,M,K} = \sqrt{I!(I+M)!(I-M)!(I+K)!(I-K)!}\tag{88}$$

to obtain [54]

$$|z_1 z_2 z_3\rangle = e^{-\frac{|u|^2}{2}} \sum_{IMK} \frac{[(2I)!]^2 z_1^{(I+M)} z_2^I z_3^{(I+K)}}{X_{I,M,K}} |IMK\rangle, \quad (89)$$

where Morales *et al.* introduced the following supplementary variable

$$|u|^2 = |z_2|^2 (1 + |z_1|^2)^2 (1 + |z_3|^2)^2. \quad (90)$$

These coherent states comply at least two requirements: continuity of labeling and resolution of unity. In relation to this latter property, we add

$$\int d\Gamma |z_1 z_2 z_3\rangle \langle z_1 z_2 z_3| = 1, \quad (91)$$

where the measure of integration $d\Gamma$ is given by [54]

$$d\Gamma = d\tau \left\{ 4[(1 + |z_1|^2)(1 + |z_3|^2)]^4 |z_2|^4 - 8[(1 + |z_1|^2)(1 + |z_3|^2)]^2 |z_2|^2 + 1 \right\} \quad (92)$$

with

$$d\tau = \frac{d^2 z_1}{\pi} \frac{d^2 z_2}{\pi} \frac{d^2 z_3}{\pi}. \quad (93)$$

In accordance with this requirement on coherent states, we can assert that the present formulation satisfy the weaker version, because the measure is non-positive definite [54].

4.2.2. Husimi function, Wehrl entropy

In order to get a valid expression for the Husimi distribution and the Wehrl entropy, a proper formulation of coherent states is essential. Using now Eq. (89) we find

$$|\langle IMK | z_1 z_2 z_3 \rangle|^2 = \frac{e^{-|u|^2}}{X_{I,M,K}^2} [(2I)!]^2 |z_1|^{2(I+M)} |z_2|^{2I} |z_3|^{2(I+K)}. \quad (94)$$

Therefore, the rotational partition function is given by

$$Z_{3D} = \sum_{I=0}^{\infty} \sum_{K=-I}^I \sum_{M=-I}^I e^{-\beta \epsilon_{I,K}}, \quad (95)$$

i.e.,

$$Z_{3D} = \sum_{I=0}^{\infty} (2I+1) e^{-I(I+1)\frac{\Theta}{T}} \sum_{K=-I}^I e^{-\left(\frac{I_{xy}}{I_z}-1\right)K^2\frac{\Theta}{T}}. \quad (96)$$

We see that Z_{2D} is recovered from Z_{3D} for the limiting case defined as the extremely prolate. The Husimi distribution yields

$$\mu(z_1, z_2, z_3) = \frac{e^{-|u|^2}}{Z_{3D}} \sum_{I=0}^{\infty} \frac{(2I)!}{I!} |v|^{2I} e^{-I(I+1)\frac{\Theta}{T}} \times g(I), \quad (97)$$

where

$$g(I) = \sum_{K=-I}^I \frac{|z_3|^{2(I+K)}}{(I+K)!(I-K)!} e^{-\left(\frac{I_{xy}}{I_z}-1\right)K^2\frac{\Theta}{T}}, \quad (98)$$

with

$$|v|^2 = (1 + |z_1|^2)^2 |z_2|^2, \quad (99)$$

$$|u|^2 = |v|^2 (1 + |z_3|^2)^2. \quad (100)$$

Other relevant property that it is easily verified for $\mu(z_1, z_2, z_3)$ is normalization in the fashion

$$\int d\Gamma \mu(z_1, z_2, z_3) = 1. \quad (101)$$

Now, we obtain the Wehrl entropy in the form

$$\mathcal{W} = \int d\Gamma \mu(z_1, z_2, z_3) \ln \mu(z_1, z_2, z_3). \quad (102)$$

The spherical rotator, that corresponds to another special case, we explicitly obtain

$$\mu(z_1, z_2, z_3) = e^{-|u|^2} \frac{\sum_{I=0}^{\infty} \frac{|u|^{2I}}{I!} e^{-I(I+1)\frac{\Theta}{T}}}{\sum_{I=0}^{\infty} (2I+1)^2 e^{-I(I+1)\frac{\Theta}{T}}}. \quad (103)$$

Having the Husimi functions the Wehrl entropy is straightforwardly computed.

In order to emphasize some special cases associated to possible applications we consider several possibilities.

1. The spherical rotator $I_{xy} = I_x = I_y = I_z$, which corresponds to $I_{xy}/I_z = 1$ (e.g. CH_4).
2. The oblate rotator $I_{xy} = I_x = I_y < I_z$, being $1/2 \leq I_{xy}/I_z < 1$ (e.g. C_6H_6).
3. The prolate rotator $I_{xy} = I_x = I_y > I_z$, thus $I_{xy}/I_z > 1$ (e.g. PCl_5).
4. The extremely prolate rotator is equivalent to the linear case (all diatomic molecules, $I_z = 0$, this is $I_{xy}/I_z \rightarrow \infty$ (e.g. CO_2 , C_2H_2)).

4.2.3. Fisher information measure

In this circumstance we define the shift invariant Fisher measure in 3D–dimensions as

$$\mathcal{F}_{3D} = \frac{1}{4} \int d\Gamma \mu(z_1, z_2, z_3) \left(\frac{\partial \ln \mu(z_1, z_2, z_3)}{\partial |u|} \right)^2. \quad (104)$$

Thus, from Eq. (97) we get

$$\varphi(z_1, z_2) = \frac{1}{2} \frac{\partial \ln \mu(z_1, z_2)}{\partial |u|} = \frac{\sum_{I=0}^{\infty} \left[\frac{|u|^{2I-1}}{(I-1)!} - \frac{|u|^{2I+1}}{(I)!} \right] e^{-I(I+1)\Theta/T}}{\sum_{I=0}^{\infty} \frac{|u|^{2I}}{(I)!} e^{-I(I+1)\Theta/T}}, \quad (105)$$

and, the corresponding Fisher measure can be expressed as

$$\mathcal{F}_{3D} = \int d\Gamma \mu(z_1, z_2) \varphi(z_1, z_2, z_3)^2 = \langle \varphi(z_1, z_2, z_3)^2 \rangle. \quad (106)$$

5. Final remarks

In this chapter, we have described some elements to motivate possible and future applications in condensed matter and information theory. Our fundamental discussion is devoted to two interesting systems, those are: the Landau diamagnetism and the rigid rotator in three dimensions. We choose these systems because the quantum mechanics is analytically solved. Specifically, the spectrum and a suitable formulation of coherent states are known without approximations.

In general, quantum distributions as the Husimi distribution, have long been seen as powerful tools for studying the quantum-classical correspondence and semi-classical aspects of quantum mechanics. Then, a crucial starting point in the present strategy, to evaluate some theoretical measures, is to get the Husimi distribution. This is made evoking a convenient set of coherent states in every system. As introduced by Gazeau and Klauder in the context of the harmonic oscillator, we use the same formal perspective of general requirements for formulations of coherent states that we use in the current contribution. Additionally, we have included some mathematical and practical details of the the present formalisms in order to make it instructive in courses of quantum mechanics (for graduates) and easy to apply to specific calculations of theoretical measures.

The present derivation of Husimi distributions is based on the evaluation of the mean value of the density operator in the basis of a single-particle coherent state. Then, after defining the Husimi distribution we are ready to make a possible semiclassical description evaluating (i) the semiclassical Wehrl entropy and (ii) the phase-space location via measures as Fisher information.

Furthermore, we evaluate the probability of observing a quantum state in a coherent state, by projecting the quantum states over the coherent states, as a function of a variable related to the coherent states. We see that the localization of probability and correspondingly the Husimi distribution in the phase space decreases as temperature increases.

As known, while the coherent states are independent-particle states, the Husimi function takes into account collective and environmental effects being necessary many wave packets of

independent-particle states to represent them. Furthermore, the thermodynamics of particles in systems does not depend on any coherent states formulation.

Finally, we remark, all results presented here were kindly obtained in an analytical fashion. We show some instances where the Landau diamagnetism is equivalent to the harmonic oscillator and, in the other example, where the linear rigid rotator is reobtained as a particular instance of the formulation in three dimensions. Some indications given in the present work lead to the conclusion that Fisher measure is a better indicator of the delocalization than Wehrl entropy.

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